

On the local regularity of very weak s -harmonic functions

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Modena, 9th September, 2025

On the occasion of the 150th of G. Vitali's Birthday

Regularity for the Δ : harmonic functions

Classical Regularity of harmonic functions

Lemma (Caccioppoli - Ciminno - Weyl)

$\Omega \subset \mathbb{R}^N$, $u \in L^1_{\text{loc}}(\Omega)$ **very weak** solution of $\Delta u = 0$, i.e.,

$$\int_{\Omega} u \Delta \varphi dx = 0 \quad \text{for any } \varphi \in C_c^\infty(\Omega)$$

$\implies u \in C^\infty(\Omega)$ and u is harmonic in Ω .

Regularity for the Δ : the Poisson equation

Sobolev regularity of solutions to Poisson equation

Theorem

$\Omega \subset \mathbb{R}^N$, $f \in L^2(\Omega)$, $u \in H^1(\Omega)$ **weak solution** to $-\Delta u = f$ in Ω , i.e.,

$$\int_{\Omega} Du \cdot Dv \, dx = \int_{\Omega} f v \, dx \quad \forall v \in H_0^1(\Omega)$$

$\implies u \in H_{\text{loc}}^2(\Omega)$ and for every $\Omega' \Subset \Omega$ we have

$$\|u\|_{H^2(\Omega')} \leq C(\Omega, \Omega') \left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right).$$

The difference quotients method

One proof of the regularity of solutions exploits the difference quotients method introduced by L. Nirenberg. Namely, setting

$$D_i^h v := \frac{v(x + he_i) - v(x)}{h}$$

for some $i = 1, \dots, N$ and $h \in \mathbb{R} \setminus \{0\}$ suff. small, one can prove that if u is a weak solution of $-\Delta u = f$ then if we put the function

$$v := -D_i^{-h} (\eta^2 D_i^h u), \eta \text{ smooth cut-off}$$

in the weak formulation of $-\Delta u = f$ one deduces that $Du \in H_{\text{loc}}^1(\Omega, \mathbb{R}^N)$.

Rmk This method applies to second-order elliptic operators with mild regularity assumptions on the coefficients and also to nonlinear equations.

The fractional Laplacian

Recall that for $w \in C_{\text{loc}}^{\gamma+2s}(\Omega) \cap L_s^1(\mathbb{R}^N)$ for some $\gamma > 0$, the **fractional Laplacian** $(-\Delta)^s u$ is defined as

$$(-\Delta)^s u(x) = c_{N,s} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,$$

for every $x \in \Omega$.

Here $C_{\text{loc}}^\sigma(\Omega)$ denotes $C^{\lfloor \sigma \rfloor}$ functions with locally $\sigma - \lfloor \sigma \rfloor$ Hölder continuous derivatives of order $\lfloor \sigma \rfloor$

$u \in L_s^1(\mathbb{R}^N)$ if

$$\|u\|_{L_s^1(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \frac{|u(x)|}{1 + |x|^{N+2s}} dx < \infty$$

Remark There are (at least!) ten equivalent ways to define $(-\Delta)^s$, see M. Kwaśnicki 2017. We mention another one.

Caffarelli-Silvestre extension approach (2007)

$(-\Delta)^s$ as a weighted Dirichlet-to Neumann operator

$$\left\{ \begin{array}{ll} \Delta_x U + \frac{1-2s}{y} U_y + U_{yy} = \frac{1}{y^{1-2s}} \operatorname{div}(y^{1-2s} \nabla_{x,y} U) = 0 & y > 0 \\ U(x, 0) = u(x) & x \in \mathbb{R}^N \\ -(-\Delta_x)^s u(x) = C_{N,s} \lim_{y \rightarrow 0^+} y^{1-2s} U_y(x, y) & x \in \mathbb{R}^N \end{array} \right.$$

More general elliptic operators are considered in place of Δ_x , by P.R. Stinga and J.L. Torrea (2010) and later also parabolic by K. Nyström and O. Sande (2016) and by P.R. Stinga and J.L. Torrea (2017).
Previous results in this circle of ideas by Muckenhoupt and Stein (1965) in harmonic analysis, Molchanov and Ostrovskii (1969) in a probabilistic setting.

Very weak solutions

In this talk for $B_1 := B_1(0)$ and $g \in L^1_s(\mathbb{R}^N)$ we deal with **very weak solutions** of

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } B_1 \\ u = g & \text{in } B_1^c, \end{cases} \quad (1)$$

i.e., $u \in L^1_s(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} u(-\Delta)^s \varphi dx = 0 \quad (2)$$

for every φ compactly supported in B_1 such that $(-\Delta)^s \varphi \in L^\infty_s(\mathbb{R}^N)$ and $u = g$ in B_1^c , where

$$L^\infty_s(\mathbb{R}^N) := \left\{ f \in L^\infty(\mathbb{R}^N); \quad \sup_{x \in \mathbb{R}^N} (1 + |x|^{N+2s}) |f(x)| < \infty \right\}.$$

Local regularity for very weak solutions via fractional difference quotients

Theorem (Carbotti, Cito, La Manna, P. 2024)

Let u be a very weak solution of $(-\Delta)^s u = 0$. Then

1. (Sobolev regularity) $u \in H_{\text{loc}}^s(B_1)$ and the estimate

$$\|u\|_{H^s(B')} \leq c(N, s, B') \|u\|_{L_s^1(\mathbb{R}^N)}$$

holds for every $B' \Subset B_1$.

2. (Classical regularity) u is real analytic in B_1 and the estimate

$$\|D^\alpha u\|_{L^\infty(B_{r_0})} \leq c^{|\alpha|} \alpha! C(R, r, N, s) \left(\|u\|_{L^\infty(B_R)} + \|u\|_{L_s^1(\mathbb{R}^N)} \right)$$

holds for any $\alpha \in \mathbb{N}_0^N$ and $0 < r_0 < R < 1$.

Available proofs of regularity

1. $(-\Delta)^s$ as a Ψ DO ([distributional solutions](#), R. Seeley, 1965)
2. Via fractional Poisson representation (Bogdan-Byczkowski, 1999)

$$u(x) = \int_{B_1^c} P_1^s(x, y) g(y) dy, \quad (3)$$

[very weak](#) solutions with $g \in C(\overline{B_1})$ or $g \in L^\infty(B_{1+\epsilon})$

3. Via mean value property (Aimar-Beltritti-Gómez 2015)
4. Via classical difference quotients (Cozzi 2017): $H_{loc}^{2s-\epsilon}$ estimates for [weak](#) solutions
5. Via maximum principle: continuity of [weak](#) solutions (Servadei-Valdinoci 2013)
6. Hölder regularity of [weak](#) solutions (Brasco-Lindgren-Schikorra 2018)
7. Via classical *difference quotients* for the extension problem (Banerjee-Garofalo 2023): analyticity of [weak](#) solutions

Sketch of our proof

Let $u \in L^1_s(B_1)$ be a very weak solution of
$$\begin{cases} (-\Delta)^s u = 0 & \text{in } B_1 \\ u = g & \text{in } B_1^c \end{cases} .$$

1. Summability improvement: $u \in L^2_{\text{loc}}(B_1)$
2. Fractional Sobolev regularity: $u \in H^s_{\text{loc}}(B_1)$
3. Higher order fractional Sobolev regularity: $u \in H^{2s}_{\text{loc}}(B_1)$
4. Hölder regularity: $u \in C^\alpha_{\text{loc}}(B_1)$
5. Improved Hölder regularity: $u \in C^{\gamma(\alpha,s)}_{\text{loc}}(B_1)$
6. Analyticity

Starting point 1

Theorem (Ros Oton-Serra 2013)

Let $s \in (0, 1)$, $N > 2s$, $m > \frac{N}{s}$, $\Omega \subset \mathbb{R}^N$ a bounded $C^{1,1}$ domain, and let v be the unique solution of

$$\begin{cases} (-\Delta)^s v = \psi & \text{in } \Omega \\ v = 0 & \text{in } \Omega^c, \end{cases} \quad (4)$$

with $\psi \in C(\overline{\Omega})$. Then, there exists $C > 0$ such that

$$\|v\|_{C^s(\mathbb{R}^N)} \leq C \|\psi\|_{L^m(\Omega)}. \quad (5)$$

Proof. Solve $(-\Delta)^s v = |\psi|$ in $\mathbb{R}^N$ via Riesz potential.

Starting point 2

Theorem (Abdellaoui-Fernández-Leonori-Younes 2023)

Let $N \geq 2$, $s \in (0, 1)$, $m \geq 1$, $\Omega \subset \mathbb{R}^N$ a bounded C^2 domain, and let v be the unique solution of

$$\begin{cases} (-\Delta)^s v = \psi & \text{in } \Omega, \\ v = 0 & \text{in } \Omega^c. \end{cases}$$

Then

$$\|v\|_{W^{s,p}(\mathbb{R}^N)} \leq C \|\psi\|_{L^m(\Omega)} \quad (6)$$

where

- ▶ if $1 \leq m < \frac{N}{s}$ then (6) holds with $1 < p < \frac{Nm}{N-ms}$
- ▶ if $m > \frac{N}{s}$ then (6) holds for every $1 < p < \infty$.

Carré du champ estimates (Γ -calculus)

$$\begin{aligned} I_s(f, g) &= \frac{1}{2} ((-\Delta)^s(fg) - f(-\Delta)^s g - g(-\Delta)^s f) \\ &= \int_{\mathbb{R}^N} \frac{(f(\cdot) - f(y))(g(\cdot) - g(y))}{|\cdot - y|^{N+2s}} dy \end{aligned}$$

Let $\eta \in C_c^\infty(B_1)$, $\text{supp } \eta = B_{1-\delta}$

$$\begin{aligned} \|I_s(\eta, \eta v)\|_{L^p(\mathbb{R}^N)} &\leq C \|\eta v\|_{W^{s,p}(\mathbb{R}^N)} \\ \|\eta I_s(\eta, v)\|_{L^p(\mathbb{R}^N)} &\leq C \left(\|v\|_{W^{s,p}(B_{1-\delta})} + \|v\|_{L_s^1(\mathbb{R}^N)} \right), \\ \|I_s(\eta, \eta v)\|_{L^\infty(\mathbb{R}^N)} &\leq C \|\eta v\|_{C^s(B_{1-\delta})} \\ [\eta I_s(\eta, v)]_{C^{\gamma(\alpha,s)}(\mathbb{R}^N)} &\leq C \left[\|v\|_{C^\alpha(B_{1-\delta})} + \|v\|_{L_s^1(\mathbb{R}^N)} \right] \\ [I_s(f, g)]_{C^{\gamma(\alpha,s)}(\mathbb{R}^N)} &\leq C[f]_{C^\alpha(\mathbb{R}^N)}[g]_{C^{0,1}(\mathbb{R}^N)} \end{aligned}$$

Remark C depend on N, s, δ, p, α .

Step 1: from L_s^1 to L_{loc}^p , $p < \frac{N}{N-s}$

$\psi \in C^\infty(B_1) \cap C(\overline{B_1})$, $\Omega := B_{1-2\delta}$, $\eta \in C_c^\infty(B_1)$ supported in $B_{1-2\delta}$.
 $(-\Delta)^s v = \psi$ as in (4) implies $(-\Delta)^s(\eta^2 v) \in L_s^\infty$, hence $\eta^2 v$ is a suitable test function and we have $\int u(-\Delta)^s(\eta^2 v) = 0$, whence

$$\begin{aligned} \left| \int_{\mathbb{R}^N} \eta^2 u \psi dx \right| &\leq \int_{\mathbb{R}^N} |u l_s(\eta, \eta v)| dx + 2 \int_{\mathbb{R}^N} |uv \eta (-\Delta)^s \eta| dx \\ &\quad + \int_{\mathbb{R}^N} |u \eta l_s(\eta, v)| dx =: A_1 + A_2 + A_3. \end{aligned}$$

$$\begin{aligned} A_2 &= \int_{\mathbb{R}^N} |uv \eta (-\Delta)^s \eta| dx \leq C \|u\|_{L^1(B_{1-2\delta})} \|v\|_{L^\infty(B_{1-2\delta})} \quad \text{by (5)} \\ &\leq C \|u\|_{L^1(B_{1-2\delta})} \|\psi\|_{L^m(B)}, \quad m > N/s \end{aligned}$$

Step 1: First summability improvement

$$\|I_s(\eta, \eta v)\|_{L_s^\infty} \leq C \|v\|_{C^s} \leq C \|\psi\|_{L^m} \quad \text{by (5)}$$

$$\text{implies } A_1 \leq C \|u\|_{L_s^1(\mathbb{R}^N)} \|\psi\|_{L^m(\mathbb{R}^N)}, \quad m > N/s.$$

$$\begin{aligned} A_3 &= \int_{\mathbb{R}^N} |u \eta I_s(\eta, v)| dx = \int_{B_{1-2\delta}} |u \eta I_s(\eta, v)| dx \\ &\leq \|u\|_{L^1(B_{1-2\delta})} \|I_s(\eta, v)\|_{L^\infty(B_{1-2\delta})} \end{aligned}$$

and

$$\begin{aligned} |I_s(\eta, v)(x)| &\leq C \|v\|_{C^s(\mathbb{R}^N)} \left(\int_{B_{2-3\delta}(x)} \frac{dy}{|x-y|^{N+s-1}} + \int_{B_\delta^c(x)} \frac{dy}{|x-y|^{N+2s}} \right) \\ &\leq C \|\psi\|_{L^m(B)} \end{aligned}$$

$$\text{implies } A_3 \leq C \|u\|_{L_s^1(\mathbb{R}^N)} \|\psi\|_{L^m(\mathbb{R}^N)}, \quad m > N/s.$$

First summability improvement: from L^p_{loc} to L^2_{loc}

We have proved that

$$\left| \int_{\mathbb{R}^N} \eta^2 u \psi dx \right| \leq C(\delta) \|u\|_{L^1_s(\mathbb{R}^N)} \|\psi\|_{L^m(B_1)}$$

for any $m > \frac{N}{s}$, $\psi \in C^\infty(B) \cap C(\overline{B})$. Therefore, by a density argument $\eta^2 u \in L^p(B_1)$ and $u \in L^p_{loc}(B_1)$ for any $p < \frac{N}{N-s}$.

Using again $\varphi = \eta^2 v$ as test function in

$$\int_{\mathbb{R}^N} u(-\Delta)^s \varphi dx = 0$$

with η, v given as before, and using the estimates provided by our Proposition and Theorem [AFly] in a finite number of steps we prove that $\eta^2 u \in L^r(B_1)$ for any $r < \frac{N}{s}$. In particular $\eta^2 u \in L^2(B_1)$ and $u \in L^2_{loc}(B_1) \cap L^1_s(\mathbb{R}^N)$.

Step 2: From $L_s^1 \cap L_{\text{loc}}^2$ to $H_{\text{loc}}^s(B_1)$

Fractional difference quotients:

Let $\delta \in (0, \frac{1}{4})$, $\tau \in (0, \frac{1}{2})$, $\eta \in C_c^\infty(B_1)$ with $\text{supp } \eta = B_{1-2\delta}$ and $\eta_\tau : [0, \infty) \rightarrow [0, 1]$ defined as

$$\eta_\tau(t) := \begin{cases} 0 & \text{if } 0 \leq t \leq \tau/2, \\ \frac{2}{\tau}t - 1 & \text{if } \tau/2 \leq t \leq \tau, \\ 1 & \text{if } t \geq \tau, \end{cases}$$

For every $x \in B_{1-\delta}$ we define the following function

$$D_{\eta_\tau, \eta}^s u(x) := \int_{\mathbb{R}^N} \eta_\tau(|x - y|) \frac{\eta(x)u(x) - \eta(y)u(y)}{|x - y|^{N+2s}} dy. \quad (7)$$

$H_{\text{loc}}^s(B_1)$ -regularity via fractional difference quotients

$u \in H_{\text{loc}}^s(B_1)$ if $\eta u \in H^s(B_1)$, where

$$\begin{aligned} [\eta u]_{H^s(\mathbb{R}^N)}^2 &= \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \frac{(\eta(x)u(x) - \eta(y)u(y))^2}{|x - y|^{N+2s}} dy \\ &= \sup_{0 < \tau \leq 1/2} \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \eta_\tau(|x - y|) \frac{(\eta(x)u(x) - \eta(y)u(y))^2}{|x - y|^{N+2s}} dy \\ &= 2 \sup_{0 < \tau \leq 1/2} \int_{\mathbb{R}^N} u \eta D_{\eta_\tau, \eta}^s u dx = \sup_{0 < \tau \leq 1/2} G_{\eta_\tau, \eta}^s(u), \end{aligned}$$

therefore we are left to estimating $G_{\eta_\tau, \eta}^s(u)$ uniformly wrto τ .

Set $\varphi := \eta v$ where v is the solution of the following problem

$$\begin{cases} (-\Delta)^s v = D_{\eta_\tau, \eta}^s u & \text{in } B_{1-\delta} \\ v = 0 & \text{in } B_{1-\delta}^c. \end{cases} \quad (8)$$

$H_{\text{loc}}^s(B_1)$ -regularity via fractional difference quotients

The function φ is a suitable test function for (2) and we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} u(-\Delta)^s \varphi \, dx = \int_{\mathbb{R}^N} u(-\Delta)^s (\eta v) \, dx \\ &= \int_{B_{1-\delta}} uv(-\Delta)^s \eta \, dx + \int_{B_{1-2\delta}} u\eta(-\Delta)^s v \, dx - \int_{\mathbb{R}^N} u l_s(\eta, v) \, dx, \end{aligned}$$

whence

$$\begin{aligned} \frac{1}{2} G_{\eta_\tau, \eta}^s(u) &= \left| \int_{B_{1-2\delta}} u\eta D_{\eta_\tau, \eta}^s u \, dx \right| \\ &\leq \left| \int_{B_{1-\delta}} uv(-\Delta)^s \eta \, dx \right| + \left| \int_{\mathbb{R}^N} u l_s(\eta, v) \, dx \right| \\ &= |C_1| + |C_2|. \end{aligned}$$

$H_{\text{loc}}^s(B_1)$ -regularity via fractional difference quotients

$$\begin{aligned} |C_1| &= \left| \int_{B_{1-\delta}} uv(-\Delta)^s \eta \, dx \right| \leq \|u\|_{L^{(2_s^*)'}(B_{1-\delta})} \|v\|_{L^{2_s^*}(\mathbb{R}^N)} \|(-\Delta)^s \eta\|_{L^\infty(\mathbb{R}^N)} \\ &\leq C(\delta)[v]_{H^s(\mathbb{R}^N)}, \quad \text{by Sobolev embedding} \end{aligned}$$

where $2_s^* = \frac{2N}{N-2s}$ and

$$\begin{aligned} C_2 &= \int_{\mathbb{R}^N} u l_s(\eta, v) dx = \int_{B_{1-\delta}} u(x) dx \int_{\mathbb{R}^N} \frac{(v(x) - v(y))(\eta(x) - \eta(y))}{|x - y|^{N+2s}} dy \\ &\quad + \int_{B_{1-\delta}^c} u(x) dx \int_{\mathbb{R}^N} \frac{(v(x) - v(y))(\eta(x) - \eta(y))}{|x - y|^{N+2s}} dy \\ &= C_3 + C_4 \end{aligned}$$

$H_{\text{loc}}^s(B_1)$ -regularity via fractional difference quotients

$$|C_3| \leq C(\eta) \|u\|_{L^2(B_{1-\delta})} [v]_{H^s(\mathbb{R}^N)}$$

$$|C_4| \leq C(\delta) \|u\|_{L_s^1(\mathbb{R}^N)} [v]_{H^s(\mathbb{R}^N)}$$

Finally, putting v as a test function in (8) we get

$$[v]_{H^s(\mathbb{R}^N)}^2 \leq [v]_{H^s(\mathbb{R}^N)} \sqrt{G_{\eta\tau, \eta}^s(u)}$$

whence

$$[\eta u]_{H^s(\mathbb{R}^N)} = \sup_{0 < \tau \leq 1/2} \sqrt{G_{\eta\tau, \eta}^s(u)} \leq C(\delta, N, s) \|u\|_{L_s^1(\mathbb{R}^N)}.$$

$$H_{\text{loc}}^s(B_1) \xrightarrow{\text{Step 3}} H_{\text{loc}}^{2s}(B_1) \xrightarrow{\text{Step 4}} C_{\text{loc}}^\alpha(B_1)$$

$u \in H_{\text{loc}}^s(B_1)$, η as before implies that $\eta^2 u$ is a **weak solution** of $(-\Delta)^s w = f \in L^2(\mathbb{R}^N)$, hence by elliptic regularity (Grubb, 2015 and Biccari-Warma-Zuazua, 2017) **$\eta^2 u \in H^{2s}(\mathbb{R}^N)$** .

Indeed, $f = 2\eta u(-\Delta)^s \eta - I_s(\eta, \eta u) - \eta I_s(\eta, u) \in L^2(\mathbb{R}^N)$ and we have already proved that all the terms are in $L^2(\mathbb{R}^N)$.

Since u is a locally weak s -harmonic function, by fractional De Giorgi estimates (Brasco-Lindgren-Schikorra 2018) we have that $u \in C_{\text{loc}}^\alpha(B_1)$ for every **$\alpha \in (0, \min\{2s, 1\})$** .

$$C_{\text{loc}}^{\alpha}(B_1) \xrightarrow{\text{Step 5}} C_{\text{loc}}^{\gamma(\alpha,s)}(B_1)$$

To reach a better Hölder regularity of u we exploit Hölder estimates on $I_s(\eta, \eta u)$ and $\eta I_s(\eta, u)$ and the fact that $\eta^2 u$ is a bounded weak solution in $\mathbb{R}^N$ to the equation $(-\Delta)^s w = f \in C^{\gamma\alpha}(\mathbb{R}^N)$. Then, by Schauder estimates, we obtain that $\eta^2 u \in C^{\gamma(\alpha,s)}(\mathbb{R}^N)$, with $\gamma(\alpha, s)$ as below. For any $f \in C^{\alpha}(\mathbb{R}^N)$ and $g \in C^{0,1}(\mathbb{R}^N)$ we have

$$[I_s(f, g)]_{C^{\gamma(\alpha,s)}(\mathbb{R}^N)} \leq C[f]_{C^{\alpha}(\mathbb{R}^N)}[g]_{C^{0,1}(\mathbb{R}^N)}$$

$$\text{where } \gamma(\alpha, s) := \begin{cases} 2\alpha - 2s & \text{if } 0 < s \leq \frac{1}{2} \\ \alpha - 2s + 1 & \text{if } \frac{1}{2} < s < 1 \end{cases}$$

Conclusion: $C_{\text{loc}}^{\gamma(\alpha,s)}(B_1) \rightarrow C^\omega(B_1)$

We conclude our proof by fixing $0 < r_0 < r < R < 1$ and using the fact that now u is regular enough to be the classical solution of

$$\begin{cases} (-\Delta)^s w = 0 & \text{in } B_r \\ w = h & \text{in } B_r^c, \end{cases} \quad (9)$$

where

$$h := \begin{cases} u & \text{in } B_1 \setminus B_r \\ g & \text{in } B_1^c. \end{cases} \in C(\overline{B_R}) \cap L_s^1(\mathbb{R}^N)$$

$$C_{\text{loc}}^{\gamma(\alpha,s)}(B_1) \rightarrow C^\omega(B_1)$$

By uniqueness

$$u(x) = u_h(x) := \int_{B_r^c} P_r^s(x,y) h(y) dy, \quad x \in B_r,$$

where

$$P_r^s(x,y) := C_{N,s} \left(\frac{r^2 - |x|^2}{|y|^2 - r^2} \right)^s \frac{1}{|x-y|^N}, x \in B_r, y \in B_r^c.$$

Moreover

$$\|u_h\|_{L^\infty(B_r)} \leq C \left(\|u\|_{L^\infty(B_R)} + \|u\|_{L_s^1(\mathbb{R}^N)} \right).$$

By simple estimates on $D_x^\alpha P_r(x,y)$ for every $\alpha \in \mathbb{N}^N$, $x \in B_{r_0}$ and $y \in B_r^c$ we finally prove the second inequality in our Main Theorem.