# Variational inequalities for the Ornstein–Uhlenbeck semigroup

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# Introduction

Let $\mu$ be a measure on $\mathbb{R}^d$ and $\{S_t\}_{t>0}$ a semigroup of operators such that $t \mapsto S_t f$ is continuous $\mu$-almost everywhere for any measurable function $f$.

To study pointwise convergence, for instance the existence of $\lim_{t\to 0+} S_t f(x)$, we usually follow a two step procedure:

1. Establish a weak type inequality:

$$\mu\Big(\Big\{x : \sup_{t>0} |S_t f(x)| > \alpha\Big\}\Big) \lesssim_p \frac{\|f\|_p^p}{\alpha^p},$$

   (which ensures that the space $\mathcal{L}^p$ of $L^p$-functions for which the limit $\lim_{t\to 0+} f$ exists $\mu$-almost everywhere is closed in $L^p(\mu)$).

2. Determine a class $\mathcal{S} \subset \mathcal{L}^p$ of functions which is dense in $L^p(\mu)$.

## Variation seminorm

We will discuss a couple of tools which allows us to handle pointwise convergence more quantitatively.

Let $1 \leqslant \varrho < \infty$. For any function $\phi$ defined on $\mathbb{R}_+ = (0, +\infty)$ the variation seminorm $\|\phi\|_{v(\varrho)}$ is defined by

$$\|\phi\|_{v(\varrho)} := \sup \left( \sum_{i=1}^{N} |\phi(t_i) - \phi(t_{i-1})|^{\varrho} \right)^{1/\varrho},$$

where the supremum is taken over all finite, increasing sequences $\{t_i\}_0^N$ in $\mathbb{R}_+$. This is a seminorm and vanishes only on constant functions.

Given a function $f \in L^p(\mathbb{R}^d)$ and a point $x \in \mathbb{R}^d$, define

$$\phi_{f,x}(t) = S_t f(x), \qquad t > 0,$$

and apply to $\phi_{f,x}$ the above definition, obtaining a function mapping $x$ to

$$\|S_\bullet f(x)\|_{v(\varrho)} = \sup \left( \sum_{i=1}^{N} |S_{t_i} f(x) - S_{t_{i-1}} f(x)|^\varrho \right)^{1/\varrho},$$

where the supremum is taken over all finite increasing sequences $\{t_i\}_0^N \subset \mathbb{R}_+$.

An important property of the function $x \mapsto \|S_\bullet f(x)\|_{v(\varrho)}$ is that if for some $x$ we have $\|S_\bullet f(x)\|_{v(\varrho)} < \infty$, then the limit $\lim_{t\to 0+} S_t f(x)$ exists (this observation was popularised by J. Bourgain).

For this reason we look for variational inequalities, that is weak (or strong) type inequalities for $f \mapsto \|S_\bullet f(\cdot)\|_{v(\varrho)}$,

$$\mu\Big(\big\{x : \|S_\bullet f(x)\|_{v(\varrho)} > \alpha\big\}\Big) \lesssim_p \frac{\|f\|_p^p}{\alpha^p}.$$

From now on, $S_t$ will be the Ornstein-Uhlenbeck semigroup.

# The Ornstein–Uhlenbeck semigroup

This is the semigroup $\left(\mathcal{H}_t\right)_{t>0}$ generated by the elliptic operator

$$\mathcal{L} = \frac{1}{2}\mathrm{tr}\left(Q\nabla^2\right) + \langle Bx, \nabla\rangle,$$

called the Ornstein-Uhlenbeck operator. Here $\nabla$ is the gradient and $\nabla^2$ the Hessian matrix. Moreover,

- $Q$ is a real, symmetric and positive definite $d \times d$ matrix, called the covariance of $\mathcal{L}$;
- $B$ is a real $d \times d$ matrix whose eigenvalues have negative real parts.

$\mathcal{H}_t$ is defined as the semigroup generated by $\mathcal{L}$,

$$\mathcal{H}_t = e^{t\mathcal{L}}, \quad t > 0.$$

For each $t > 0$, $\mathcal{H}_t$ is an integral operator. It is well known that there exists a measure $\gamma_\infty$ which is invariant under the action of $\mathcal{H}_t$, that is

$$\int_{\mathbb{R}^d} \mathcal{H}_t f(x) d\gamma_\infty(x) = \int_{\mathbb{R}^d} f(x) d\gamma_\infty(x) \quad \text{for all } f \text{ and } t > 0.$$

This measure, which is unique up to a positive factor, is given by

$$d\gamma_\infty(x) = (2\pi)^{-\frac{d}{2}} (\det Q_\infty)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\langle Q_\infty^{-1} x, x\rangle\right) dx,$$

where $Q_\infty$ is the positive matrix given by

$$Q_\infty = \int_0^\infty e^{sB} Q e^{sB^*} ds.$$

Then, for each $f \in L^1(\gamma_\infty)$ and all $t > 0$ one has

$$\mathcal{H}_t f(x) = \int K_t(x,u) f(u) d\gamma_\infty(u), \quad x \in \mathbb{R}^d,$$

where

$$K_t(x,u) = \left(\frac{\det Q_\infty}{\det Q_t}\right)^{1/2} e^{\langle Q_\infty^{-1} x, x\rangle/2} \times \exp\left[-\frac{1}{2}\left\langle \left(Q_t^{-1} - Q_\infty^{-1}\right)(u - D_t x), u - D_t x\right\rangle\right]$$

is the Mehler kernel. Here

- $Q_t = \int_0^t e^{sB} Q e^{sB^*} ds, \quad 0 < t \leqslant \infty;$
- $D_t = Q_\infty e^{-tB^*} Q_\infty^{-1}$.

The Ornstein–Uhlenbeck operator $\mathcal{L}$ and the Ornstein–Uhlenbeck semigroup $\mathcal{H}_t$ play the role of the Laplacian and of the heat semigroup in $\mathbb{R}^d$ if the Lebesgue measure $dx$ is replaced by $d\gamma_\infty$.

In general, the semigroup $\mathcal{H}_t$ is not self-adjoint, (not even normal), and to study its properties we cannot rely on the spectral theorem. Since, for each $t > 0$, $\mathcal{H}_t$ is an integral operator, our analysis is instead based on the properties of the Mehler kernel.

Recently, V. Almeida, J. J. Betancor, J. C. Farina, P. Quijano and L. Rodriguez-Mesa in *Littlewood–Paley functions associated with general Ornstein–Uhlenbeck semigroups*, proved the following result:

**Theorem:** For each $\varrho > 2$ and $1 < p < \infty$, the operator mapping $f \in L^p(\gamma_\infty)$ to

$$\|\mathcal{H}_\bullet f(x)\|_{v(\varrho)}, \quad x \in \mathbb{R}^d,$$

where the $v(\varrho)$ seminorm is taken in the variable $t$, is bounded from $L^p(\gamma_\infty)$ to $L^p(\gamma_\infty)$.

*A bit of history:* In fact, in 2001 R. L. Jones and K. Reinhold proved that for $\varrho > 2$ the variation operator of any symmetric diffusion semigroup is bounded on $L^p$ for $1 < p < \infty$. Ten years later C. Le Merdy and Q. Xu extended this result to a non-symmetric context. Almeida et al. applied the latter result in the Gaussian, non-symmetric framework, that we are discussing.

We completed the picture proving the endpoint result:

**Theorem**

For each $\varrho > 2$ the operator that maps $f \in L^1(\gamma_\infty)$ to the function

$$\mathbb{R}^d \ni x \mapsto \|\mathcal{H}_\bullet f(x)\|_{v(\varrho)}$$

is of weak type $(1,1)$ with respect to the measure $\gamma_\infty$.
For $1 \leqslant \varrho \leqslant 2$ the assertion is false.

We have to prove the inequality

$$\gamma_\infty\big(\{x \in \mathbb{R}^d : \|\mathcal{H}_\bullet f(x)\|_{v(\varrho),\mathbb{R}_+} > \alpha\}\big) \lesssim \frac{\|f\|_{L^1(\gamma_\infty)}}{\alpha}$$

for all $\alpha > 0$ and all $f \in L^1(\gamma_\infty)$.
We first proved it for $d = 1$ with a method which seems hard to adapt to the case $d > 1$ (Ann. Mat. Pura Appl. (2024)).
Then recently, with a different method, we proved this estimate also for $d > 1$ (to appear in J. Geom. Anal.)).
Anyway, in order to avoid some technicalities, we shall present the recent proof (that holding for all $d \geqslant 1$) in the one dimensional case.

If $d = 1$ the semigroup is given by

$$\mathcal{H}_t f(x) = \int_{-\infty}^{\infty} K_t(x,u)\, f(u)\, d\gamma_\infty(u), \quad t > 0,$$

where $f \in L^1(\gamma_\infty)$, $K_t$ is the Mehler kernel

$$K_t(x,u) = \frac{e^{x^2/2}}{\sqrt{1-e^{-2t}}} \exp\left(-\frac{1}{2}\frac{(e^{-t}u - x)^2}{1-e^{-2t}}\right), \quad (x,u) \in \mathbb{R} \times \mathbb{R},$$

and $d\gamma_\infty(u) = (2\pi)^{-1/2} \exp(-u^2/2) du$.

Let $\varrho > 2$. The proof of

$$\gamma_\infty\big(\{x \in \mathbb{R} : \|\mathcal{H}_\bullet f(x)\|_{v(\varrho)}\big) > \alpha\} \lesssim \frac{1}{\alpha}\|f\|_{L^1(\gamma_\infty)},$$

is divided into several steps. First of all we distinguish between small and large $t$.

The reason to distinguish between small and large times is due to the different behaviour of the Mehler kernel in these two cases.

In addition, when $t$ is small we shall need another (spatial) distinction, between local and global regions. We start discussing the case $t \geqslant 1$.

# The case of large $t$

The easiest case is the variation of $\mathcal{H}_t f(x)$ for $1 \leqslant t < \infty$.

## Proposition

*For each* $\varrho \geqslant 1$ *the operator that maps* $f \in L^1(\gamma_\infty)$ *to the function*

$$\|\mathcal{H}_\bullet f(x)\|_{v(\varrho),[1,+\infty)}, \quad x \in \mathbb{R},$$

*is of weak type* $(1,1)$ *with respect to the measure* $\gamma_\infty$*. In fact, one has the following stronger result:*

$$\gamma_\infty\left(\left\{x \in \mathbb{R} : \|\mathcal{H}_\bullet f(x)\|_{v(\varrho),[1,\infty)} > \alpha\right\}\right) \lesssim \frac{1}{\alpha\sqrt{\log \alpha}} \|f\|_{L^1(\gamma_\infty)}, \quad \alpha > 2.$$

This estimate, which is enhanced by a logarithmic factor, is optimal. An analogous phenomenon was already observed both for the Ornstein–Uhlenbeck maximal operator and for the Gaussian Riesz transform.

# Proof:

**Remark:** If $\phi \in C^1(I)$ and $\phi' \in L^1(I)$, then $\|\phi\|_{v(\varrho),I} < \infty$; in fact,

$$\|\phi\|_{v(\varrho),I} \leqslant \int_I |\dot{\phi}(t)| dt.$$

Hence,

$$\begin{aligned}
\|\mathcal{H}_\bullet f(x)\|_{v(\varrho),[1,\infty)} &\leqslant \int_1^\infty |\partial_t \mathcal{H}_t f(x)| \, dt \\
&\leqslant \int_1^\infty \left| \partial_t \int_{-\infty}^\infty K_t(x,u) f(u) d\gamma_\infty(u) \right| \, dt \\
&\leqslant \int_{-\infty}^\infty \left( \int_1^\infty \left| \dot{K}_t(x,u) \right| dt \right) |f(u)| d\gamma_\infty(u).
\end{aligned}$$

We are thus reduced to bound the integral

$$\int_1^\infty \left|\dot{K}_t(x,u)\right| dt.$$

From the expression of the Mehler kernel

$$K_t(x,u) = \frac{e^{x^2/2}}{\sqrt{1-e^{-2t}}} \exp\left(-\frac{1}{2}\frac{(e^{-t}u-x)^2}{1-e^{-2t}}\right) \quad ,(x,u)\in\mathbb{R}\times\mathbb{R},$$

it is easy to deduce

$$\int_1^\infty \left|\dot{K}_t(x,u)\right| dt \lesssim e^{x^2/2}.$$

The last bound implies

$$\|\mathcal{H}_\bullet f(x)\|_{v(\varrho),[1,\infty)} \leqslant \int_{\mathbb{R}} \left( \int_1^\infty \left| \dot{K}_t(x,u) \right| dt \right) |f(u)| d\gamma_\infty(u) \lesssim e^{x^2/2} \|f\|_1.$$

Since

$$\gamma_\infty\Big(\big\{x : e^{x^2/2} > \alpha\big\}\Big) \lesssim \frac{1}{\alpha\sqrt{\log\alpha}}, \quad \alpha > 2,$$

we get

$$\gamma_\infty\left(\left\{x \in \mathbb{R} : \|\mathcal{H}_\bullet f(x)\|_{v(\varrho),[1,\infty)} > \alpha\right\}\right) \lesssim \frac{1}{\alpha\sqrt{\log\alpha}} \|f\|_{L^1(\gamma_\infty)}, \quad \alpha > 2,$$

which yields the result.

The estimate

$$\gamma_\infty\Big(\{x \in \mathbb{R} : e^{|x|^2/2} > \alpha\}\Big) \lesssim \frac{1}{\alpha\sqrt{\log \alpha}}, \quad \alpha > 2,$$

is not true for $d > 1$. To prove the claimed bound we need to introduce a suitable system of polar coordinates adapted to the problem.

# The case of small $t$

The study of the variation in $(0, 1]$ is more delicate and requires a further distinction between local and global parts of the Ornstein-Uhlenbeck semigroup operator.
To do that we first split $\mathbb{R}^2$ into a local and a global region. This decomposition was introduced for $d = 1$ by B. Muckenhoupt in the 70's and for $\mathbb{R}^d \times \mathbb{R}^d$ with $d > 1$ by P. Sjögren in the 80's.

**The local region** is the region where $|x-u| \lesssim 1/(1+|x|)$ The point of this definition is that in the local region the density of the measure $\gamma_\infty$ has the same order of magnitude at $x$ and $u$.

The **local part** of the semigroup is then defined by

$$\mathcal{H}_t^{\mathrm{loc}} f(x) = \int f(u) K_t(x,u) \eta\big((1+|x|)|x-u|\big)\, d\gamma_\infty(u),$$

where $\eta$ is a smooth nonnegative function on $\mathbb{R}_+$ which is 1 in $(0,1/2]$ and 0 in $[1,\infty)$.
The **global part** $\mathcal{H}_t^{\mathrm{glob}} = \mathcal{H}_t - \mathcal{H}_t^{\mathrm{loc}}$ is given by a similar expression, with $\eta(.)$ replaced by $1-\eta(\cdot)$.

# The global case for small $t$

The operator is

$$\mathcal{H}_t^{\mathrm{glob}} f(x) = \int K_t(x,u)\Big(1-\eta((1+|x|)|x-u|)\Big) f(u)\, d\gamma_\infty(u).$$

Its kernel is supported where $|x-u| \gtrsim 1/(1+|x|)$.

## Proposition

*For each $\varrho \geqslant 1$ the operator that maps $f \in L^1(\gamma_\infty)$ to the function $x \mapsto \|\mathcal{H}_\bullet^{\mathrm{glob}} f(x)\|_{v(\varrho),(0,1]}$ satisfies*

$$\gamma_\infty\left(\left\{x \in \mathbb{R} : \|\mathcal{H}_\bullet^{\mathrm{glob}} f(x)\|_{v(\varrho),(0,1]} > \alpha\right\}\right) \lesssim \frac{1}{\alpha}\|f\|_{L^1(\gamma_\infty)}, \quad \alpha > 2.$$

# Sketch of the proof:

As before we have

$$
\begin{aligned}
\|\mathcal{H}_{\bullet}^{\mathrm{glob}} f(x)\|_{v(\varrho),(0,1]} &\leqslant \int_0^1 \left|\partial_t \mathcal{H}_t^{\mathrm{glob}} f(x)\right| dt \\
&= \int_0^1 \left|\partial_t \int_{-\infty}^{\infty} K_t(x,u)(1-\eta) f(u) d\gamma_\infty(u)\right| dt \\
&= \int_0^1 \left|\int_{-\infty}^{\infty} \dot{K}_t(x,u)(1-\eta) f(u) d\gamma_\infty(u)\right| dt \\
&\leqslant \int_{-\infty}^{\infty} \left(\int_0^1 \left|\dot{K}_t(x,u)\right| dt\right)(1-\eta)|f(u)| d\gamma_\infty(u).
\end{aligned}
$$

It is easy to see that $\dot{K}_t(x,u)$ has at most four zeros in $(0,1)$. Denote them by $t_1,\ldots,t_{N-1}$ ($N$ and $t_i$ may depend on $(x,u)$ and $N\leqslant 5$). Set also $t_0=0$ and $t_N=1$. Then

$$\int_0^1 |\dot{K}_t(x,u)|\,dt = \sum_1^{N(x,u)} \int_{t_{i-1}(x,u)}^{t_i(x,u)} |\dot{K}_t(x,u)|dt$$

$$= \sum_1^{N(x,u)} \left| \int_{t_{i-1}(x,u)}^{t_i(x,u)} \dot{K}_t(x,u)\,dt \right| \leqslant 10 \sup_{(0,1]} K_t(x,u).$$

Hence,

$$\|\mathcal{H}^{\mathrm{glob}}_{\bullet} f(x)\|_{v(\varrho),(0,1]} \leqslant \int_{-\infty}^{\infty}\int_0^1 \left|\dot{K}_t(x,u)\right| dt\,(1-\eta)|f(u)|\,d\gamma_\infty(u)$$

$$\leqslant 10 \int_{-\infty}^{\infty} \sup_{(0,1]} K_t(x,u)\left(1-\eta((1+|x|)|x-u|)\right)\Big)|f(u)|\,d\gamma_\infty(u).$$

We claim that for all $(x, u)$

$$\sup_{(0,1]} K_t(x,u)\Big(1-\eta((1+|x|)|x-u|)\Big) \lesssim e^{x^2/2}\,(1+|x|).$$

This yields

$$\|\mathcal{H}_{\bullet}^{\mathrm{glob}} f(x)\|_{v(\varrho),(0,1]} \lesssim e^{x^2/2}\,(1+|x|)\int_{-\infty}^{\infty} |f(u)|\, d\gamma_\infty(u),$$

which implies the result, because

$$\gamma_\infty\left(\left\{x : e^{x^2/2}\,(1+|x|) > \alpha\right\}\right) \lesssim \frac{1}{\alpha}, \quad \alpha > 0.$$

Again, for $d > 1$ the estimate

$$\gamma_\infty\left(\left\{x : e^{x^2/2}\left(1+|x|\right) > \alpha\right\}\right) \lesssim \frac{1}{\alpha}, \quad \alpha > 0.$$

is no longer true.
When $d > 1$, we again use that in the global region $\dot{K}_t(x,u)$ has a bounded number $N$, say, of zeros in $(0,1]$, obtaining

$$\begin{aligned}\|\mathcal{H}_t^{\mathrm{glob}} f(x)\|_{v(\varrho),(0,1]} \\ \leqslant 2N \int_{\mathbb{R}^d} \sup_{t\in(0,1]} K_t(x,u)\left(1-\eta(x,u)\right)|f(u)|\,d\gamma_\infty(u),\end{aligned}$$

and then that the maximal operator (with sup inside the integral)

$$Mf(x) = \int \sup_{t\in(0,1]} K_t(x,u)\left(1-\eta(x,u)\right)|f(u)|d\gamma_\infty(u)$$

is of weak type $(1,1)$ with respect to $\gamma_\infty$.

# The local case for small $t$

The operator is

$$\mathcal{H}_t^{\mathrm{loc}} f(x) = \int f(u)\, K_t(x,u)\, \eta((1+|x|)|x-u|)\, d\gamma_\infty(u),$$

living where

$$|x-u| \leqslant 1/(1+|x|).$$

**Theorem**

*For each $\varrho > 2$ the operator mapping $f \in L^1(\gamma_\infty)$ to*

$$x \mapsto \|\mathcal{H}_\bullet^{\mathrm{loc}} f(x)\|_{v(\varrho),(0,1]},$$

*is of weak type $(1,1)$ with respect to $\gamma_\infty$.*

Here $\varrho > 2$, in fact the estimate is false for $\varrho \leqslant 2$.

# Splitting the line in local intervals

In this case $\mathcal{H}_t^{\mathrm{loc}} f(x)$ depends only on the restriction of $f$ to the interval $\{u : |u-x| \leqslant 1/(1+|x|)\}$. We thus split the line into intervals of similar type, choosing an increasing sequence $\{x_j\}_0^\infty$ with $x_0 = 0$ and

$$x_{j+1} - \frac{1}{1+x_{j+1}} = x_j + \frac{1}{1+x_j}$$

for $j = 1, \dots$ ($x_j \approx 2\sqrt{j} - 1$) and setting $x_j = -x_{|j|}$ for $j < 0$. Defininig

$$\mathbb{I}_j = \left[ x_j - \frac{1}{1+|x_j|},\, x_j + \frac{1}{1+|x_j|} \right],$$

we get the decompositions: $\mathbb{R} = \bigcup_{j \in \mathbb{Z}} \mathbb{I}_j$.

When $d > 1$, we have an analogous decomposition with the intervals replaced by rings.

**Claim**: If

$$\operatorname{supp} f \subset \mathbb{I}_j = \left[x_j - \frac{1}{1+|x_j|},\, x_j + \frac{1}{1+|x_j|}\right],$$

then

$$\operatorname{supp} \mathcal{H}_t^{\mathrm{loc}} f \subset \widetilde{\mathbb{I}}_j = \left[x_j - \frac{4}{1+|x_j|},\, x_j + \frac{4}{1+|x_j|}\right], \qquad j \in \mathbb{Z}.$$

Since the intervals $\widetilde{\mathbb{I}}_j$ have the bounded overlapping property we may therefore assume that $\operatorname{supp} f \subset \mathbb{I}_j$.

In each $\widetilde{\mathbb{I}}_j$ the density of $\gamma_\infty$ is essentially constant,

$$e^{-x^2/2} \approx e^{-x_j^2/2} \quad \text{for } x \in \widetilde{\mathbb{I}}_j,$$

and the implicit constants are uniform in $j$. We can therefore replace on each interval $\gamma_\infty$ with the Lebesgue measure and apply vector-valued Calderón–Zygmund theory to prove that for each $\varrho > 2$ the operator mapping a function $f \in L^1(\gamma_\infty)$ supported in $\mathbb{I}_j$ to

$$x \mapsto \|\mathcal{H}_\bullet^{\mathrm{loc}} f(x)\|_{v(\varrho),(0,1]},$$

is of weak type $(1,1)$.

We call $\Theta$ the set of finite strictly increasing sequences $\underline{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_{N(\underline{\varepsilon})})$ in $(0,1]$ and $F$ the space of functions on $\Theta \times \mathbb{N}$. Then we introduce the subspace $F_\varrho$ of functions $g$ for which

$$\|g\|_{F_\varrho} := \sup_{\underline{\varepsilon}\in\Theta} \left( \sum_{j=0}^{N(\underline{\varepsilon})} \left| g(\underline{\varepsilon}, j) \right|^\varrho \right)^{1/\varrho} < \infty,$$

this is a norm and $F_\varrho$ is a Banach space.
Given a function $f$ on $\mathbb{R}$, we define $Vf : \mathbb{R} \to F$ by

$$\big(Vf(\underline{\varepsilon}, j)\big)(x) = \mathcal{H}^{\mathrm{loc}}_{\varepsilon_j} f(x) - \mathcal{H}^{\mathrm{loc}}_{\varepsilon_{j-1}} f(x), \quad j = 1, \ldots, N(\underline{\varepsilon}).$$

Then

$$\|\mathcal{H}^{\mathrm{loc}}_{\bullet} f(x)\|_{v(\varrho),(0,1]} = \|\big(Vf(\cdot,\cdot)\big)(x)\|_{F_\varrho}.$$

Instead of proving that the operator mapping $f \in L^1(\gamma_\infty)$ to

$$x \mapsto \|\mathcal{H}_\bullet^{\mathrm{loc}} f(x)\|_{v(\varrho),(0,1]}$$

is of weak type $(1,1)$ with respect to $\gamma_\infty$, we can prove:

**Theorem**

*For each $\varrho > 2$ the operator that maps $f \in L^1(\gamma_\infty)$ to*

$$x \mapsto \|Vf(x)\|_{F_\varrho}, \quad x \in \mathbb{R},$$

*is of weak type $(1,1)$ with respect to $\gamma_\infty$.*

To apply Calderón–Zygmund theory we have to pass to the Lebesgue measure, hence, we define

$$\widetilde{V}g(x) = e^{-x^2/2} V\left(g(\cdot)e^{x^2/2}\right)(x).$$

If the operator that maps $g \in L^1(du)$ to

$$x \mapsto \left\| \widetilde{V}g(x) \right\|_{F_\varrho}, \quad x \in \mathbb{R},$$

is of weak type $(1,1)$ with respect to the Lebesgue measure

$$\Downarrow$$

The operator that maps $f \in L^1(\gamma_\infty)$ to

$$x \mapsto \|Vf(x)\|_{F_\varrho}, \quad x \in \mathbb{R},$$

is of weak type $(1,1)$ with respect to $\gamma_\infty$.

To prove that $\widetilde{V}$ is of weak type $(1,1)$ we express it as an integral operator

$$\widetilde{V}g(x) = \int_{\mathbb{R}} \widetilde{M}(x,u)g(u)du,$$

where $\widetilde{M}(x,u) \in F$. For $x \notin \operatorname{supp} g$ this is a Bochner integral in $F_\varrho$ and $\widetilde{V}g(x) \in F_\varrho$.

Since the variation operator for $\mathcal{H}_t^{\mathrm{loc}}$ is bounded on $L^2(\gamma_\infty)$ for $\varrho > 2$ (Le Merdy-Xu), the operator

$$g \mapsto \left\| \widetilde{V}g(\cdot) \right\|_{F_\varrho}$$

is bounded on $L^2(dx)$.

It suffices therefore to show that $\widetilde{M}$ is a standard kernel:

(a)

$$\|\widetilde{M}(x,u)\|_{F_\varrho} \lesssim \frac{1}{|x-u|}, \quad x \neq u,$$

(b) if moreover $|x-u| > 2|u-u'|$, then

$$\|\widetilde{M}(x,u) - \widetilde{M}(x,u')\|_{F_\varrho} \lesssim \frac{|u-u'|}{|x-u|^2}.$$

From these bounds and the boundedness of $g \mapsto \left\|\widetilde{V}g(\cdot)\right\|_{F_\varrho}$ on $L^2(dx)$ we finally obtain the required weak type $(1,1)$ estimate, concluding the proof.